EXPLORATION OF VIJAY KRISHNA'S CONJECTURE FOR MATRIX GAMES

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Exploration of Vijay Krishna's Conjecture for Matrix Games

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Abstract

This paper studies a conjecture posed by Vijay Krishna on whether every joint probability distributions can be realized as a correlated equilibrium of some matrix game. If the probability distribution maintains full rank and non-zero row and column sums, then conjecture states that one can generate a matrix game for which the observed distribution is a correlated equilibrium.

I prove the correctness of the conjecture for the cases of size 1×3 and 2×2. Numerical simulations suggest the conjecture is true for 3×3 games. This approximate verification is achieved through reformulations of the problem as various constrained optimization problems.

This paper also explores a game reduction strategy and constraint relaxation strategy for developing a general induction argument. Although these methodologies fail to generalize, they elucidate the behavior of the conjecture.

Contents

1 Introduction

Game Theory is the mathematical study of interactive decision making. As such, it has a variety of important applications in the social sciences. This thesis is devoted to the study of a conjecture about properties of a particular kind of game formulated by Vijay Krishna. Some of the required mathematical machinery for its study will be described here, but this paper will not serve as a self-contained body of work.

There are many types of games: simultaneous-move games, sequential games, games of incomplete information, and more. However, this study focuses exclusively on matrix games, which are defined in Chapter 2. In Chapters 3 and 4, I formulate Vijay Krishna's conjecture, discuss and analyze the exploratory findings of the research conducted over the course of the study period, and present alternative formulations and adjustments that hope to guide future approaches. Lastly, in Chapter 5, I discuss some of computational proofs that were completed when considering each of the approaches undertaken in Chapter 4.

2 Definitions

2.1 Matrix Games

Definition 2.1. *Matrix Game*

A *Matrix Game* is an interaction between a set $N = \{1, \dots, N\}$ players where each player i has a strategy space A_i and a payoff u_i. Each player i's payoff is a map

u_i: $A_i \to \mathbb{R}$. If each player $i \in N$ plays $a_i \in A_i$, then, the payoff to player $i \in N$ is $u_i \times (a_1, \ldots, a_n)$. Matrix Games are particular games with $|N| = 2$ and finite action spaces $|A_1|, |A_2| < \infty$.

Matrix games can be represented in matrix form where the rows are labelled with the strategies for the row player (player 1) and the columns are labelled with the strategies for the column player (player 2). The $(i, j)^{th}$ entry of the matrix is $[u_1(a_i, a_j), u_2(a_i, a_j)]$.

2.2 Solution Concepts

In order to motivate the conjecture itself, I define the most pertinent solution concepts and discuss their relationships, using formulations by Gibbons and Vohra et al. as reference [1] [2].

Definition 2.2. *Mixed and Pure Strategies*

The elements of $A_i = \{a_{i1}, \ldots, a_{im}\}$ are called the *pure strategies* available to player i. A *mixed* strategy for player i is a probability distribution $p_i = (p_{i1}, \ldots, p_{im})$ $p_i \in \Delta A_i$ where of course the laws of probability are satisfied: $0 \leq p_{i1} \leq 1$ and $\sum_{i=1}^{m} p_{ik} = 1 \; \forall i \in N$. ΔA_i denotes the set of probability vectors over action space A_i.

Note that the set of pure strategies in a given action space A_i of a player i is properly contained in the set of mixed strategies. Take $p_i = \{0, \ldots, p_{ij} = 1, \ldots, 0\}$ which would correspond to player i's choosing of strategy j with probability 1. This is true $\forall j \in \{1, \ldots, m\}$, $\forall i \in N$.

Definition 2.3. *Strategy Profile*

A *strategy profile* in a matrix game is a list of actions chosen by each player:

$$a = (a_1, \ldots, a_N)$$

If all $a_1, \ldots, a_N$ are pure strategies, then a is a *pure* strategy profile. Otherwise, a is a *mixed* strategy profile. Let a_{-i} denote the action profile of all players excluding player i.

Continuing, I now define both Pure Strategy and Mixed Strategy Nash Equilibria. These solutions capture steady-state behavior in games when players have no incentive to deviate unilaterally.

Definition 2.4. *Pure Strategy Nash Equilibrium*

The pure strategy profile $(a_1^*, \ldots, a_m^*)$ is a *Pure Strategy Nash Equilibrium* if for each player i in the player space N, a_i^* is a best response to a_{-i}. Explicitly,

$$u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i}) \;\forall i \in N \;\forall a_i^* \in A_i$$

Definition 2.5. *Mixed Strategy Nash Equilibrium*

The mixed strategy profile $(p_1^*, \ldots, p_N^*)$ is a *Mixed Strategy Nash Equilibrium* if the following holds:

$$\sum_{j=1}^{m} p_{ij}^* u_i(a_{ij}, a_{-i}) \geq \sum_{j=1}^{m} p_{ij} u_i(a_{ij}, a_{-i}) \quad \forall p_{ij} \in \Delta A_i \quad \forall i \in N$$

Intuitively, a Nash Equilibrium is a set of actions such that no player stands to gain from unilaterally deviating from his/her action when fixing every other player's actions. In the case of Mixed Strategy Equilibria, the inequality holds in expectation over a mixed strategy profile. In 1951, John Nash of course proved their existence in [3] for games with finite action spaces.

Nash Equilibria (NE) are well-known, well-understood, and extremely powerful tools in game theory. A natural extension of this tool are Correlated Equilibria (CE). Existence of CE in matrix games with finite action spaces has been proven by Hart and Schmeidler, but initially suggested as a solution concept by Aumann [4] [5]. Less ubiquitous than NE, CE are useful in situations where players may benefit from correlating their actions with one another. In certain scenarios, players may benefit from this by increasing overall wellfare. A public signal that tells each player what mixed strategy they should employ, which allows for correlation of actions amongst players. I will show the conditions for existence of correlated equilibria hold since they are a generalization of NE [6].

Definition 2.6. *Correlated Equilibrium*

A probability distribution P over pure action profiles a is a *Correlated Equilibrium* if no player rationally wishes to deviate from the public signal's recommendation. For each player i, and every public signal a_i^*:

$$E_{a,P}[u_i(a_i^*, a_{-i})|a_i^*] \geq E_{a,P}[u_i(a_i, a_{-i})|a_i^*] \; \forall i \in N \; \forall a_i \in A_i \setminus \{a_i^*\}$$

where $E_{\delta,\rho}[x]$ is defined as the expected value of x with respect to a distribution ρ where δ is drawn from ρ.

2.3 Farkas' Lemma

To consider the dual formulation of the system generated by the conjecture, I employ Farkas' Lemma, which was developed in 1902 by Julius Farka and reformulated in [7]. This Lemma allows us to solve an equivalent system. In the dual formulation of the problem, there are only equality constraints. Below is the aforementioned Lemma in addition to a diagram that guides geometric interpretation.

Definition 2.7. *Farkas' Lemma*

Let A be an m x n matrix, $b \in \mathbb{R}^m$ and $F = \{x \in \mathbb{R}^n : Ax = b\ , x \geq 0\}$. Either $F \neq \emptyset$ or $\exists y \in \mathbb{R}^m$ such that $yA \geq 0$ and $yb < 0$ but not both.

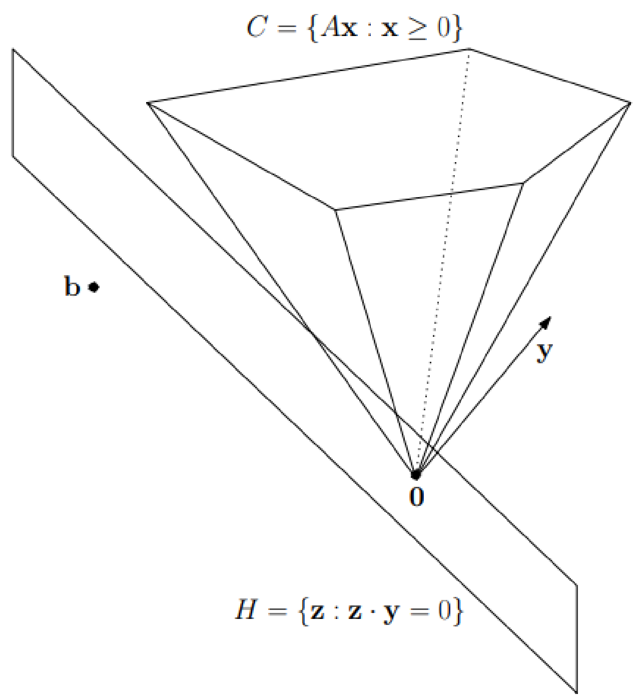

Figure 1: Geometric Interpretation of Farkas' Lemma [8]

This Lemma is best understood by considering the polyhedral cone C that contains all the solutions to the primal formulation of a system of equations. Either

$\exists s \in C$ a solution to the primal contained in the collection of points C, or $\nexists y \in C$ a solution to the dual formulation. Equivalently, either $b \in C$ or there is a hyperplane H separating the polyhedral cone C from $\{b\}$ where $0 \in H, y \perp H$.

2.4 Feasibility of Systems of Equations

Below I review facts about linear programming and linear algebra that will be utilized in this paper.

Definition 2.8. *Extreme Point*

Consider a convex region $R \subseteq \mathbb{R}^n$ where $n \geq 1$. An *extreme point* $e \in R$ is a point e that cannot be expressed as a convex combination of any other points in R.

Definition 2.9. *Linear Program (LP)*

A *LP* is an optimization problem that satisfies the following properties:

1. The goal is to maximize or minimize a linear function of decision variables
2. The constraints the objective function are subject to are affine
3. Each decision variable x_i satisfies one of the three following properties:
   (a) ≥ 0
   (b) ≤ 0
   (c) unrestricted in sign

Definition 2.10. *Primal and Dual of an LP*

Consider A, b, c where $A \in M_{m,n}(\mathbb{R})$, $b \in M_{m,1}(\mathbb{R})$, $c \in M_{1,n}(\mathbb{R})$. Then the primal LP (and its corresponding dual) are as follows:

$$\text{Primal: } \max c^T x \text{ s.t. } Ax \leq b, x \geq 0$$

$$\text{Dual: } \min y^T b \text{ s.t. } y^T A = c, y \geq 0$$

The relationship between the primal and dual are defined by the Weak and Strong Duality Theorems, formulated below similarly that of Van Roy et al. [9].

Theorem 2.11. *Weak Duality Theorem*

Consider $x, y \in \mathbb{R}^m$ feasible points for the primal and dual, respectively. Then:

$$c^T x \leq y^T A x \leq b^T y$$

This finding shows that unboundedness in the primal implies infeasiblity in the dual. Similarly, unboundedness in the dual implies infeasible in the primal. From this theorem follows the Strong Duality Theorem.

Theorem 2.12. *Strong Duality Theorem*

If either the primal or dual has a finite optimal solution, then so does the other. Specifically, the optimal values are equal, and there exist optimal solutions to both the primal and the dual.

3 Problem Formulation

3.1 Vijay Krishna's Conjecture

Given any matrix and a joint probability distribution over its entries, can this distribution be interpreted as a correlated equilibrium of some matrix game? Vijay Krishna's conjecture is principally motivated by this question.

Conjecture 3.1. *For any joint probability matrix for a two player game — that satisfies non-zero row and column sums and that maintains full rank — there exists a matrix game for which that joint probability matrix is a correlated equilibrium.*

Let R be the set of pure strategies for the row player and C the set of strategies for the column player. Let p_{ij} be the probability that row should play $i \in R$ and column should play $j \in C$. The goal is to show that there exists a payoff matrix $W = \{w_{ij}\}$ such that the conditions for a correlated equilibrium are satisfied. I will refer to the corresponding system of inequalities as PRIM.

PRIM

$$\sum_{j \in C} w_{i,j} * p_{i,j} > \sum_{j \in C} w_{r,j} * p_{i,j} \forall i \in R, \forall r \in R \setminus \{i\}$$

$$\sum_{i \in R} w_{i,j} * p_{i,j} > \sum_{i \in R} w_{i,r} * p_{i,j} \forall j \in C, \forall r \in C \setminus \{i\}$$

Strict inequality must be imposed on the various p_{ij}'s because otherwise the system would have no solution. Supposing you permitted equality, then the following inequality could occur, which is impossible:

$$\sum_{j \in C} w_{i,j} > \sum_{j \in C} w_{r,j} > \sum_{j \in C} w_{i,j} \forall i \in R, \forall r \in R \setminus \{i\}$$

Assuming the matrix P of p_{ij}'s is of full rank ameliorates this issue. Additionally, assume that each of the row and column sums are non-zero because otherwise, a particular row (or column) action would weakly dominate the other actions of that row (or column) and the game could then be reduced to a subgame with non-zero row and column sums.

Invoking Farkas' Lemma, I construct an equivalent system of equations (called ALT) that exhibits the following relationship to the original system of equations. PRIM has a solution iff ALT does not have a solution.

ALT

$$p_{ij} \sum_{r \in R \setminus \{i\}} x_{ir} - \sum_{r \in R \setminus \{i\}} p_{rj} x_{ri} + p_{ij} \sum_{k \in C \setminus \{j\}} y_{jk} - \sum_{k \in C \setminus \{j\}} p_{ik} y_{kj} = 0 \; \forall i \in R, j \in C$$

$$\sum_{i \in R} \sum_{r \in R \setminus \{i\}} x_{ir} + \sum_{j \in C} \sum_{k \in C \setminus \{i\}} y_{jk} = 1$$

$$x_{ij}, y_{ij} \geq 0 \; \forall i \in R, j \in C$$

4 Exploratory Analysis

4.1 Empirical Verification in the Primal

The first step taken when exploring the behavior of the conjecture empirically was to verify its truth for small joint probability matrices. A MATLAB script was written to generate randomly a joint probability matrix of size 2×2 and size 3×3. It was generated repeatedly until it satisfied the properties of non-zero column and row sums and was of full rank. Then a game matrix for which that joint probability matrix is a CE was generated. Practically, the only complication with verification of the conjecture using PRIM is that a system of strict inequalities doesn't permit all of the well-defined operations permitted for systems of the form $Ax = b$.

To account for this complication in verifying the conjecture using PRIM, the r.h.s of the system below was perturbed by $\epsilon > 0, \epsilon \in \mathbb{R}$ such that the conditions could still be satisfied, but the system would instead be a collection of equality constraints. This system, called ϵ-perturbed PRIM, was then reformulated as a constrained optimization problem. The adjusted system is shown below. The conjecture was verified using this approach for both 2×2 and 3×3 cases.

ϵ-perturbed PRIM

$$\min \ f = \sum_{i=1}^{k} |w_i| \ k \in \{2, 3\}$$

$$s.t.$$

$$\left[A\right] * \begin{bmatrix} w_{12} \\ w_{21} \\ . \\ . \\ . \\ w_{jj} \end{bmatrix} \geq \begin{bmatrix} \epsilon \\ \epsilon \\ \epsilon \\ . \\ . \\ . \\ \epsilon \end{bmatrix}$$

$$\sum_{i=1}^{k} w_i = 1$$

In this case I define $Ax \geq b$ as the inequality holding true entry-wise for each constraint in the system. Here $[A]$ is either of size 4×4 or 12×9 for $k = 2, 3$ respectively, and is the matrix representation of the coefficients of PRIM. The above objective function was chosen to attempt to push the solution towards a simpler or interpretable game.

4.1.1 Empirical Verification in the Dual

Empirical verification in the dual formulation of the system of equations is favorable since it leads to a system of equality constraints. Again, Farka's Lemma is used to convert the system of equations into its dual formulation. As a reminder, ALT is shown below.

$$p_{ij} \sum_{r \in R \setminus \{i\}} x_{ir} - \sum_{r \in R \setminus \{i\}} p_{rj} x_{ri} + p_{ij} \sum_{k \in C \setminus \{j\}} y_{jk} - \sum_{k \in C \setminus \{j\}} p_{ik} y_{kj} = 0 \ \forall i \in R, j \in C$$

$$\sum_{i \in R} \sum_{r \in R \setminus \{i\}} x_{ir} + \sum_{j \in C} \sum_{k \in C \setminus \{i\}} y_{jk} = 1$$

$$x_{ij}, y_{ij} \geq 0 \ \forall i \in R, j \in C$$

Empirically, in both 2×2 and 3×3 cases, there did not exist x, y such that all of the equality constraints were satisfied for any randomly generated probability matrix. This implies that PRIM has a solution, and so the conjecture is verified empirically. In order to learn more from this observed behavior, ALT was then reconstructed as an LP. This procedure guided the research of this conjecture insofar as it demonstrated by how many constraints (roughly) the system seemed to violate on average. This approach is described in 4.3. Having said that, ALT is shown below.

ALT

$$[B] \begin{bmatrix} x_{12} \\ x_{13} \\ y_{21} \\ . \\ . \\ . \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x_{ij}, y_{ij} \geq 0 \quad \forall i \in R \quad \forall j \in C$$

$[B]$ denotes the matrix representation of ALT (size 10×12 in the case of generating a joint probability matrix of size 3×3). This system can be solved directly in MATLAB without requiring any reformulation. Doing so verified the conjecture since every generated 2×2 and 3×3 probability distribution empirically led to a system without a solution, which implies that PRIM has a solution.

4.2 Reformulation using Unrestricted Variables

The reformulation of the ALT as an LP with artificial variables always has a solution. Artificial variables provide an infeasibility certificate for ALT by allowing the violation of constraints to be expressed through increases in the artificial variables corresponding to that constraint. These artificial variables were added to the system in the following way. Each constraint i in the ALT was given an unrestricted variable in the form of two artificial variables S_i, Z_i s.t. $S_i, Z_i \geq 0, S_i, Z_i \in \mathbb{R}$ $\forall\ S_i \in [S]\ Z_i \in [Z]$. The new LP for the 3×3 case is shown below.

U.r.s ALT

$$\min\ f = 0 * (x_{12} + x_{13} + \ldots) + M * (\sum_{i=1}^{10} S_i) + N * (\sum_{i=1}^{10} Z_i)$$

$$s.t.$$

$$[B]\begin{bmatrix} x_{12} \\ x_{13} \\ y_{21} \\ y_{13} \\ . \\ . \\ . \end{bmatrix} + [I]\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ . \\ . \\ . \\ S_{10} \end{bmatrix} - [I] * \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ . \\ . \\ . \\ Z_{10} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x_{ij}, y_{ij} \geq 0 \quad \forall i \in R = \{1,2,3\} \quad \forall j \in C = \{1,2,3\}$$
$$S_i, Z_i \geq 0\ \forall i \in \{1,2,\ldots,10\}$$

To be clear, each constraint i has a corresponding S_i, Z_i, and each identity matrix is obviously square of size 10. Additionally, M, N are arbitrarily large real-valued numbers.

Different cases for M, N were considered. 100 was chosen as an arbitrarily large value for empirical analysis. Additionally, only behavior from the 3×3 will be considered below.

1. $M = N = 1$
2. $M = N = 100$
3. $M > N; M = 100, N = 1$
4. $M < N; N = 100, M = 1$

In Case 1, observe that the optimal solution usually exhibited at most two non-zero S_i and one non-zero Z_i. This same behavior was observed in Case 2 up to scaling. There were no observed patterns with regards to which constraints on average exhibited non-zero S_i or Z_i but removing two constraints from the first nine leads to a solution with no non-zero S_i or Z_i. The last constraint was excluded in that case since removing the last constraint leads to an unbounded solution.

In Case 3, the same pattern described in Cases 1 and 2 is observed. However, in Case 4, $Z_1 = 1, Z_2 = Z_3 = \ldots = Z_{10} = 0$ and each $S_i > 0$ but small. This is asymmetry can probably be attributed to some sort of tie-breaking mechanism in MATLAB's functionality as I expect Case 3 and 4 to be quite similar. Having said that, this warrants further investigation as it is plausible that this asymmetry can be attributed to some property of the U.r.s. ALT.

4.3 Constraint Relaxation

Theoretically, if you took a relaxation of ALT, call it RALT, and show its infeasibility, then it would follow that ALT itself is infeasible. If one could construct RALT in such a way that the RALT for the n case properly contained the size-$(n-1)$ case's RALT, then by induction one could show that the conjecture were true in this way. This was the motivation for this approach.

Consider ALT in the 3×3 where certain constraints are relaxed. One such system is shown below.

<u>RALT</u>

$$\begin{aligned}
P_1(x_{12} + x_{13}) &= P_2 x_{21} + P_3 x_{31} \\
P_2(x_{21} + x_{23}) &= P_1 x_{12} + P_3 x_{32} \\
P^1(y_{12} + y_{13}) &= P^2 x_{21} + P^3 y_{31} \\
P^2(y_{21} + y_{23}) &= P^1 x_{12} + P^3 y_{32} \\
\sum_{i \in R} \sum_{r \in R \setminus \{i\}} x_{ir} + \sum_{j \in C} \sum_{k \in C \setminus \{i\}} y_{jk} &= 1 \\
x_{ij}, y_{ij} \geq 0 \ \forall i \in R, \forall j &\in C
\end{aligned}$$

The system above is generated from ALT by summing the first three equations, summing the second three equations, and summing the first, fourth, and seventh equations. This approach may appear ad-hoc, but this RALT was chosen to gain intuition for the behavior of how relaxations of certain constraints impacted feasibility. In fact, the sum of the first nine equations of ALT is zero, which implies that at least one of those nine constraints is redundant. It turns out that removing more than one constraints leads to a RALT that is in fact feasible, and so it became difficult to find a RALT that could work for the described induction argument.

4.4 Game Reduction to Infeasible Subgames

To prove the 2×3 case, instead employ the facts that the ALTs corresponding to the 1×3 and 2×2 cases are infeasible (proven in 5.1 and 5.2). From symmetry the ALTs corresponding to every 1×3, 3×1, and 2×2 are infeasible. The goal of this approach is to show by contradiction that the ALT corresponding to the 2×3 case cannot be feasible which implies that the conditions for a CE are satisfied.

It is worth noting that this approach fails since setting any of the variables to zero in ALT of the the 2×3 case other than $x_{23} = x_{32} = 0$ doesn't lead to a system that corresponds to removing a singleton or more from either R or C. This prevents this strategy from being fruitful.

Consider the ALT of the case where one observes the 2×3 probability matrix of the game: $R = \{2, 3\}, C = \{1, 2, 3\}$. Suppose for contradiction the system has a solution (and therefore that the conjecture is false). Below is the system in the form $Ax = b$.

$$\begin{bmatrix} p_{21} & -p_{31} & p_{21} & p_{21} & -p_{22} & 0 & -p_{23} & 0 \\ p_{22} & -p_{32} & -p_{21} & 0 & p_{22} & p_{22} & 0 & -p_{23} \\ p_{23} & -p_{33} & 0 & -p_{21} & 0 & -p_{22} & p_{23} & p_{23} \\ -p_{21} & p_{31} & p_{31} & p_{31} & -p_{32} & 0 & -p_{33} & 0 \\ -p_{22} & p_{32} & -p_{31} & 0 & p_{32} & p_{32} & 0 & -p_{33} \\ -p_{23} & p_{33} & 0 & -p_{31} & 0 & -p_{32} & p_{33} & p_{33} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} * \begin{bmatrix} x_{23} \\ x_{32} \\ y_{12} \\ y_{13} \\ y_{21} \\ y_{23} \\ y_{31} \\ y_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Suppose we are maximizing some arbitrary objective function $c \cdot x$ subject to the system above.

$$\max c \cdot x \text{ s.t. } Ax = b, x_i \geq 0 \ \forall x_i \in \vec{x}$$

Each constraint is affine, so the feasible region in consideration is a polyhedron, which is convex and closed. Therefore all optimal points are extreme points. The sum of the first six equations is zero which implies that the null-space is non-trivial or that the rank of the system isn't full. It follows that there are two non-basic variables in the system.

Suppose $x_{23} = x_{32} = 0$. Then the system above reduces to the system corresponding to the game $R = \{2\}, C = \{1, 2, 3\}$ which is in-feasible since every 1×3 is infeasible (proven in 5.2). Suppose WLOG $x_{23} = 0, x_{32} > 0$. Then if one sums the first three rows and first two columns it implies that x_{23} is a linear multiple of x_{32}:

$$(p_{21} + p_{22} + p_{23})x_{23} - (p_{31} + p_{32} + p_{33})x_{32} = 0 \implies x_{23} = C * x_{32}, C \in \mathbb{R}$$

If one of x_{23}, x_{32} is zero so is the other, and then the system reduces to the ALT of the 1×3 game, which is infeasible. Therefore both x_{23}, x_{32} are strictly positive.

The hope with this approach is that one could show that removing any two variables from the system above resulted in an infeasible system. Reducing the 2×3 game to a 1×3 corresponds to removing rows and columns from the original system (in addition to variables). Below is an example of how this approach fails.

Consider the 2×2 game $R = \{2, 3\}, C = \{1, 2\}$. It's ALT is below.

$$\begin{bmatrix} p_{21} & -p_{31} & p_{21} & -p_{22} \\ p_{22} & -p_{32} & -p_{21} & p_{22} \\ -p_{21} & p_{31} & p_{31} & -p_{32} \\ -p_{22} & p_{32} & -p_{31} & p_{32} \\ 1 & 1 & 1 & 1 \end{bmatrix} * \begin{bmatrix} x_{23} \\ x_{32} \\ y_{12} \\ y_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

To reduce the 2×3 game to this sub-game, one must exclude $y_{23}, y_{32}, y_{31}, y_{13}$, remove rows 3 and 6, and remove columns $4, 6, 7$, and 8. This can be understood visually by striking out those rows and columns from the 2×3 system. This striking-out procedure is shown below. The intuition is that removing these variables removes the opportunity for the row player to deviate from action 2 to action 1, from action 3 to action 1, and vice versa.

$$\begin{bmatrix} p_{21} & -p_{31} & p_{21} & \cancel{p_{21}} & -p_{22} & \cancel{0} & \cancel{-p_{23}} & \cancel{0} \\ p_{22} & -p_{32} & -p_{21} & \cancel{0} & p_{22} & \cancel{p_{22}} & \cancel{0} & \cancel{-p_{23}} \\ \cancel{p_{23}} & \cancel{-p_{33}} & \cancel{0} & \cancel{-p_{21}} & \cancel{0} & \cancel{-p_{22}} & \cancel{p_{23}} & \cancel{p_{23}} \\ -p_{21} & p_{31} & p_{31} & \cancel{p_{31}} & -p_{32} & \cancel{0} & \cancel{-p_{33}} & \cancel{0} \\ -p_{22} & p_{32} & -p_{31} & \cancel{0} & p_{32} & \cancel{p_{32}} & \cancel{0} & \cancel{-p_{33}} \\ \cancel{-p_{23}} & \cancel{p_{33}} & \cancel{0} & \cancel{-p_{31}} & \cancel{0} & \cancel{-p_{32}} & \cancel{p_{33}} & \cancel{p_{33}} \\ 1 & 1 & 1 & \cancel{1} & 1 & \cancel{1} & \cancel{1} & \cancel{1} \end{bmatrix} * \begin{bmatrix} x_{23} \\ x_{32} \\ y_{12} \\ \cancel{y_{13}} \\ y_{21} \\ \cancel{y_{23}} \\ \cancel{y_{31}} \\ \cancel{y_{32}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cancel{0} \\ 0 \\ 0 \\ \cancel{0} \\ 1 \end{bmatrix}$$

This example indicates that the fact that the 2×3 has two non-basic variables is not enough to prove infeasibility. Since both $x_{23}, x_{32} > 0$, two of the remaining variables must be zero. Setting any two of the six remaining variables does not result in a subgame as the example above demonstrates since one must remove four variables to arrive at a 2×2 subgame and both x_{23}, x_{32} to result in a 1×3 subgame.

Attempting to prove the 3×3 case using this methodology fails in a similar way. The corresponding ALT has three non-basic variables since one equation is redundant, but any 2×3 or 3×2 corresponds to removing four variables (and the corresponding rows and columns) from ALT. This fact makes knowing that there are three non-basic variables not useful when proving infeasibility by contradiction.

5 Small Cases of the Conjecture

In the following section we show the validity of the conjecture in the case that P is size 2×2 and 1×3. These cases were computed with the intended use of applying them to the game reduction strategy described in Chapter 4.

The following cases described were computed using Gaussian elimination. We use the following notation to denote row and column sums of the observed probability matrix P. Let P_i denote row i sum from the joint probability matrix $\forall i$. Let P^j denote column j sum from the joint probability matrix $\forall j$.

5.1 Proof of the 2×2 Case

In the 2×2 case we have $|R| = |C| = 2$.

Additionally, we assume that the matrix P of $p_{i,j}$'s is of full rank and has non-zero row and column sums:

$$c_1 * [p_{1,1}, p_{2,1}] + c_2 * [p_{1,2}, p_{2,2}] = 0 \iff c_1 = c_2 = 0$$

$$p_{11} + p_{12} > 0, p_{21} + p_{22} > 0, p_{11} + p_{21} > 0, p_{12} + p_{22} > 0$$

If there were a row and/or sum with zero sum, we could reduce the game to a smaller matrix where this wasn't true. WLOG assume reduction was performed.

Again, we need to show that there exists $\{w_{i,j}\}$'s s.t. the conditions for a CE are satisfied:

$$\sum_{j \in C} w_{i,j} * p_{i,j} > \sum_{j \in C} w_{r,j} * p_{i,j} \forall i \in R, \forall r \in R \setminus \{i\}$$

$$\sum_{i \in R} w_{i,j} * p_{i,j} > \sum_{i \in R} w_{i,r} * p_{i,j} \forall j \in C, \forall r \in C \setminus \{i\}$$

From Farkas' Lemma, we know that solving this system of equations is equivalent to showing that the system below (ALT) does not have a solution.

$$p_{ij} \sum_{r \in R \setminus \{i\}} x_{ir} - \sum_{r \in R \setminus \{i\}} p_{rj} x_{ri} + p_{ij} \sum_{k \in C \setminus \{j\}} y_{jk} - \sum_{k \in C \setminus \{j\}} p_{ik} y_{kj} = 0 \ \forall i \in R, j \in C$$

$$\sum_{i \in R} \sum_{r \in R \setminus \{i\}} x_{ir} + \sum_{j \in C} \sum_{k \in C \setminus \{j\}} y_{jk} = 1$$

In the 2×2 case, ALT can be represented in matrix form as such:

$$\begin{bmatrix} p_{11} & -p_{21} & p_{11} & -p_{12} \\ p_{12} & -p_{22} & -p_{11} & p_{12} \\ -p_{11} & p_{21} & p_{21} & -p_{22} \\ -p_{12} & p_{22} & -p_{21} & p_{22} \\ 1 & 1 & 1 & 1 \end{bmatrix} * \begin{bmatrix} x_{12} \\ x_{21} \\ y_{12} \\ y_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

We can now solve directly.

$$\left[\begin{array}{cccc|c} p_{11} & -p_{21} & p_{11} & -p_{12} & 0 \\ p_{12} & -p_{22} & -p_{11} & p_{12} & 0 \\ -p_{11} & p_{21} & p_{21} & -p_{22} & 0 \\ -p_{12} & p_{22} & -p_{21} & p_{22} & 0 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$(\xrightarrow{R1=R3+R1})(\xrightarrow{R4=R4+R2})$$

$$\left[\begin{array}{cccc|c} p_{11} & -p_{21} & p_{11} & -p_{12} & 0 \\ p_{12} & -p_{22} & -p_{11} & p_{12} & 0 \\ 0 & 0 & P^1 & -P^2 & 0 \\ 0 & 0 & -P^1 & P^2 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$(\xrightarrow{R4=R4+R3})$$

$$\left[\begin{array}{cccc|c} p_{11} & -p_{21} & p_{11} & -p_{12} & 0 \\ p_{12} & -p_{22} & -p_{11} & p_{12} & 0 \\ 0 & 0 & P^1 & -P^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

Case 1: Assume $p_{11} > 0$.

$$(\xrightarrow{R1=\frac{R1}{p_{11}}})(\xrightarrow{R2=R2-p_{12}*R1})(\xrightarrow{R5=R5-R1})$$

$$\left[\begin{array}{cccc|c} 1 & -\frac{p_{21}}{p_{11}} & 1 & -\frac{p_{12}}{p_{11}} & 0 \\ 0 & -p_{22}+\frac{p_{22}p_{21}}{p_{11}} & -P_1 & \frac{p_{12}P_1}{p_{11}} & 0 \\ 0 & 0 & P^1 & -P^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1+\frac{p_{21}}{p_{11}} & 0 & 1+\frac{p_{12}}{p_{11}} & 1 \end{array}\right]$$

P has full rank $\implies p_{22}p_{11} \neq p_{21}p_{12} \implies -p_{22}+\frac{p_{22}p_{21}}{p_{11}} \neq 0$

$$(\xrightarrow{R2=\frac{R2}{-p_{22}+\frac{p_{22}p_{21}}{p_{11}}}})$$

$$\left[\begin{array}{cccc|c} 1 & -\frac{p_{21}}{p_{11}} & 1 & -\frac{p_{12}}{p_{11}} & 0 \\ 0 & 1 & \frac{-p_{11}P_1}{(p_{21}-p_{11})p_{22}} & \frac{p_{12}P_1}{(p_{21}-p_{11}))p_{22}} & 0 \\ 0 & 0 & P^1 & -P^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1+\frac{p_{21}}{p_{11}} & 0 & 1+\frac{p_{12}}{p_{11}} & 1 \end{array}\right]$$

$$(\xrightarrow{R5=R5-(1+\frac{p_{21}}{p_{11}})R2})$$

$$\left[\begin{array}{cccc|c} 1 & -\frac{p_{21}}{p_{11}} & 1 & -\frac{p_{12}}{p_{11}} & 0 \\ 0 & 1 & \frac{-p_{11}P_1}{(p_{21}-p_{11})p_{22}} & \frac{p_{12}P_1}{(p_{21}-p_{11})p_{22}} & 0 \\ 0 & 0 & P^1 & -P^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{P^1P_1}{(p_{21}-p_{22})p_{22}} & \frac{P_1}{p_{11}} - \frac{P^1P_1p_{12}}{p_{11}p_{22}(p_{21}-p_{11})} & 1 \end{array}\right]$$

$$\left(\xrightarrow{R3=\frac{R3}{P^1}}\right)$$

$$\left[\begin{array}{ccccc|c} 1 & -\frac{p_{21}}{p_{11}} & 1 & & -\frac{p_{12}}{p_{11}} & 0 \\ 0 & 1 & \frac{-p_{11}P_1}{(p_{21}-p_{11})p_{22}} & & \frac{p_{12}P_1}{(p_{21}-p_{11})p_{22}} & 0 \\ 0 & 0 & 1 & & -\frac{P^2}{P^1} & 0 \\ 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & \frac{P^1P_1}{(p_{21}-p_{22})p_{22}} & \frac{P_1}{p_{11}} - & \frac{P^1P_1p_{12}}{p_{11}p_{22}(p_{21}-p_{11})} & 1 \end{array}\right]$$

$$\left(\xrightarrow{R5=R5-\frac{P^1P_1}{(p_{21}-p_{22})p_{22}}R3}\right)$$

$$\left[\begin{array}{ccccc|c} 1 & -\frac{p_{21}}{p_{11}} & 1 & -\frac{p_{12}}{p_{11}} & 0 \\ 0 & 1 & \frac{-p_{11}P_1}{(p_{21}-p_{11})p_{22}} & \frac{p_{12}P_1}{(p_{21}-p_{11})p_{22}} & 0 \\ 0 & 0 & 1 & -\frac{P^2}{P^1} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{P_1}{p_{11}} - \frac{P^1P_1p_{12}}{p_{11}p_{22}(p_{21}-p_{11})} + \frac{P^2P_1}{(p_{21}-p_{22})p_{22}} & 1 \end{array}\right]$$

$$\left(\xrightarrow{R4=R4+R5}\right)\left(\xrightarrow{R5=R5+R4}\right)$$

$$\left[\begin{array}{cccc|c} 1 & -\frac{p_{21}}{p_{11}} & 1 & -\frac{p_{12}}{p_{11}} & 0 \\ 0 & 1 & \frac{-p_{11}P_1}{(p_{21}-p_{11})p_{22}} & \frac{p_{12}P_1}{(p_{21}-p_{11})p_{22}} & 0 \\ 0 & 0 & 1 & -\frac{P^2}{P^1} & 0 \\ 0 & 0 & 0 & \alpha_1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

where $\alpha_1 = \frac{P_1[p_{22}(p_{21}-p_{11})-P^1p_{12}]}{p_{11}p_{22}(p_{21}-p_{11})} + \frac{P^2P_1}{(p_{21}-p_{22})p_{22}}$

$$\left(\xrightarrow{R4=\frac{R4}{\alpha_1}}\right)$$

$$\left[\begin{array}{cccc|c} 1 & -\frac{p_{21}}{p_{11}} & 1 & -\frac{p_{12}}{p_{11}} & 0 \\ 0 & 1 & \frac{-p_{11}P_1}{(p_{21}-p_{11})p_{22}} & \frac{p_{12}P_1}{(p_{21}-p_{11})p_{22}} & 0 \\ 0 & 0 & 1 & -\frac{P^2}{P^1} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{\alpha_1} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

If $\frac{1}{\alpha_1} = 0$, then $\implies y_{21} = y_{12} = x_{21} = x_{12} = 0 \Rightarrow\Leftarrow$

If $\frac{1}{\alpha_1} < 0$, then $y_{21} < 0 \Rightarrow\Leftarrow$

If $\frac{1}{\alpha_1} > 0$, then $x_{21} = \frac{p_{11}P_1P^2 - p_{12}P_1^2}{(p_{21}-p_{11})p_{22}P_1\alpha_1} < 0 \Rightarrow\Leftarrow \implies$ Therefore in **Case 1**, ALT has no solution.

Case 2: Assume $p_{12} > 0$.

$$\xrightarrow{R2=\frac{R2}{p_{12}}}$$

$$\left[\begin{array}{cccc|c} p_{11} & -p_{21} & p_{11} & -p_{12} & 0 \\ 1 & -\frac{p_{22}}{p_{12}} & -\frac{p_{11}}{p_{12}} & 1 & 0 \\ 0 & 0 & P^1 & -P^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$(\xrightarrow{R5=R5-R2})(\xrightarrow{R1=R1-p_{11}R2})$$

$$\left[\begin{array}{cccc|c} 0 & -p_{21}+\frac{p_{22}p_{11}}{p_{12}} & \frac{P_1p_{11}}{p_{12}} & -P_1 & 0 \\ 1 & -\frac{p_{22}}{p_{12}} & -\frac{p_{11}}{p_{12}} & 1 & 0 \\ 0 & 0 & P^1 & -P^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{P^2}{p_{12}} & \frac{P_1}{p_{12}} & 0 & 1 \end{array}\right]$$

$$(\xrightarrow{R1=R1+R2})(\xrightarrow{R2=R2-R1})$$

$$\left[\begin{array}{cccc|c} 1 & -p_{21}+\frac{p_{22}p_{11}-p_{22}}{p_{12}} & \frac{P_1p_{11}-p_{11}}{p_{12}} & -P_1+1 & 0 \\ 0 & p_{21}-\frac{p_{22}p_{11}}{p_{12}} & \frac{-P_1p_{11}}{p_{12}} & P_1 & 0 \\ 0 & 0 & P^1 & -P^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{P^2}{p_{12}} & \frac{P_1}{p_{12}} & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R2=\frac{R2}{p_{21}-\frac{p_{22}p_{11}}{p_{12}}}}$$

$$\left[\begin{array}{cccc|c} 1 & -p_{21}+\frac{p_{22}p_{11}-p_{22}}{p_{12}} & p_{11}+\frac{p_{11}^2-p_{11}}{p_{12}} & -P_1+1 & 0 \\ 0 & 1 & \frac{-p_{11}P_1}{p_{21}p_{12}-p_{22}p_{11}} & \frac{P_1p_{12}}{p_{21}p_{12}-p_{22}p_{11}} & 0 \\ 0 & 0 & P^1 & -P^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{P^2}{p_{12}} & \frac{P_1}{p_{12}} & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R3=\frac{R3}{P^1}}$$

$$\left[\begin{array}{cccc|c} 1 & -p_{21}+\frac{p_{22}p_{11}-p_{22}}{p_{12}} & p_{11}+\frac{p_{11}^2-p_{11}}{p_{12}} & -P_1+1 & 0 \\ 0 & 1 & \frac{-p_{11}P_1}{p_{21}p_{12}-p_{22}p_{11}} & \frac{P_1p_{12}}{p_{21}p_{12}-p_{22}p_{11}} & 0 \\ 0 & 0 & 1 & -\frac{P^2}{P^1} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{P^2}{p_{12}} & \frac{P_1}{p_{12}} & 0 & 1 \end{array}\right]$$

$$\left(\xrightarrow{R5=R5-(\frac{P^2}{p_{12}})R2}\right)$$

$$\left[\begin{array}{cccc|c}
1 & -p_{21}+\frac{p_{22}p_{11}-p_{22}}{p_{12}} & p_{11}+\frac{p_{11}^2-p_{11}}{p_{12}} & -P_1+1 & 0 \\
0 & 1 & \frac{-p_{11}P_1}{p_{21}p_{12}-p_{22}p_{11}} & \frac{P_1p_{12}}{p_{21}p_{12}-p_{22}p_{11}} & 0 \\
0 & 0 & 1 & -\frac{P^2}{P^1} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{P_1}{p_{12}}+\frac{P^2}{p_{12}}\frac{p_{11}P_1}{p_{21}p_{12}-p_{22}p_{11}} & -\frac{P_1P^2}{p_{21}p_{12}-p_{22}p_{11}} & 1
\end{array}\right]$$

$$\left(\xrightarrow{R5=R5-(\frac{P_1}{p_{12}}+\frac{P^2}{p_{12}}\frac{p_{11}P_1}{p_{21}p_{12}-p_{22}p_{11}})R3}\right)$$

$$\left[\begin{array}{cccc|c}
1 & -p_{21}+\frac{p_{22}p_{11}-p_{22}}{p_{12}} & p_{11}+\frac{p_{11}^2-p_{11}}{p_{12}} & -P_1+1 & 0 \\
0 & 1 & \frac{-p_{11}P_1}{p_{21}p_{12}-p_{22}p_{11}} & \frac{P_1p_{12}}{p_{21}p_{12}-p_{22}p_{11}} & 0 \\
0 & 0 & 1 & -\frac{P^2}{P^1} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_2 & 1
\end{array}\right]$$

where $\alpha_2 = -\frac{P_1P^2}{p_{21}p_{12}-p_{22}p_{11}} + (\frac{P_1}{p_{12}} + \frac{P^2}{p_{12}}\frac{p_{11}P_1}{p_{21}p_{12}-p_{22}p_{11}})(\frac{P^2}{P^1})$

α_2 is non-zero since the determinant of P is non-zero, p_{12} is non-zero by assumption, and each row and column sum is non-zero.

$$\left(\xrightarrow{R5=\frac{R5}{\alpha_2}}\right)$$

$$\left[\begin{array}{cccc|c}
1 & -p_{21}+\frac{p_{22}p_{11}-p_{22}}{p_{12}} & p_{11}+\frac{p_{11}^2-p_{11}}{p_{12}} & -P_1+1 & 0 \\
0 & 1 & \frac{-p_{11}P_1}{p_{21}p_{12}-p_{22}p_{11}} & \frac{P_1p_{12}}{p_{21}p_{12}-p_{22}p_{11}} & 0 \\
0 & 0 & 1 & -\frac{P^2}{P^1} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \frac{1}{\alpha_2}
\end{array}\right]$$

$$\left(\xrightarrow{R4=R4+R5}\right)\left(\xrightarrow{R5=R5-R4}\right)$$

$$\left[\begin{array}{cccc|c}
1 & -p_{21}+\frac{p_{22}p_{11}-p_{22}}{p_{12}} & p_{11}+\frac{p_{11}^2-p_{11}}{p_{12}} & -P_1+1 & 0 \\
0 & 1 & \frac{-p_{11}P_1}{p_{21}p_{12}-p_{22}p_{11}} & \frac{P_1p_{12}}{p_{21}p_{12}-p_{22}p_{11}} & 0 \\
0 & 0 & 1 & -\frac{P^2}{P^1} & 0 \\
0 & 0 & 0 & 1 & \frac{1}{\alpha_2} \\
0 & 0 & 0 & 0 & 0
\end{array}\right]$$

Similar to **Case 1**, this system also has no solution.

This system of equations is inconsistent and therefore has no solution. Invoking Farkas' Lemma, this implies that our original system of equations has a solution.

$\Longrightarrow \exists \{w_{i,j}\}'s$ such that the conditions for a CE are satisfied. □

5.2 Proof of 1×3 Case

The 1×3 case is simple in the sense that its ALT has no x's present in the system. Just as in the 2×2 case, if we show that ALT has no solution, this implies that the original formulation has a solution – that the conditions for a correlated equilbrium are satisfied. Here is ALT for the 1×3 case. WLOG suppose $R = \{1\}, C = \{1, 2, 3\}$.

$$\begin{bmatrix} p_{11} & p_{11} & -p_{12} & 0 & -p_{13} & 0 \\ -p_{11} & 0 & p_{12} & p_{12} & 0 & -p_{13} \\ 0 & -p_{11} & 0 & -p_{12} & p_{13} & p_{13} \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} * \begin{bmatrix} y_{12} \\ y_{13} \\ y_{21} \\ y_{23} \\ y_{31} \\ y_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cccccc|c} p_{11} & p_{11} & -p_{12} & 0 & -p_{13} & 0 & 0 \\ -p_{11} & 0 & p_{12} & p_{12} & 0 & -p_{13} & 0 \\ 0 & -p_{11} & 0 & -p_{12} & p_{13} & p_{13} & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$(\xrightarrow{R2=R2+R1})$$

$$\left[\begin{array}{cccccc|c} p_{11} & p_{11} & -p_{12} & 0 & -p_{13} & 0 & 0 \\ 0 & p_{11} & 0 & p_{12} & -p_{13} & -p_{13} & 0 \\ 0 & -p_{11} & 0 & -p_{12} & p_{13} & p_{13} & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$(\xrightarrow{R3=R3+R2})$$

$$\left[\begin{array}{cccccc|c} p_{11} & p_{11} & -p_{12} & 0 & -p_{13} & 0 & 0 \\ 0 & p_{11} & 0 & p_{12} & -p_{13} & -p_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

Since each column sum must be non-zero, this implies that $p_{11} > 0$.

$$(\xrightarrow{R1=\frac{R1}{p_{11}}})(\xrightarrow{R1=\frac{R2}{p_{11}}})$$

$$\left[\begin{array}{cccccc|c} 1 & 1 & -\frac{p_{12}}{p_{11}} & 0 & -\frac{p_{13}}{p_{11}} & 0 & 0 \\ 0 & 1 & 0 & \frac{p_{12}}{p_{11}} & -\frac{p_{13}}{p_{11}} & -\frac{p_{13}}{p_{11}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\left(\xrightarrow{R4=R4-R1}\right)$$

$$\left[\begin{array}{cccccc|c} 1 & 1 & -\frac{p_{12}}{p_{11}} & 0 & -\frac{p_{13}}{p_{11}} & 0 & 0 \\ 0 & 1 & 0 & \frac{p_{12}}{p_{11}} & -\frac{p_{13}}{p_{11}} & -\frac{p_{13}}{p_{11}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+\frac{p_{12}}{p_{11}} & 1 & 1+\frac{p_{13}}{p_{11}} & 1 & 1 \end{array}\right]$$

$$\left(\xrightarrow{R3=R3+R4}\right)$$

$$\left[\begin{array}{cccccc|c} 1 & 1 & -\frac{p_{12}}{p_{11}} & 0 & -\frac{p_{13}}{p_{11}} & 0 & 0 \\ 0 & 1 & 0 & \frac{p_{12}}{p_{11}} & -\frac{p_{13}}{p_{11}} & -\frac{p_{13}}{p_{11}} & 0 \\ 0 & 0 & 1+\frac{p_{12}}{p_{11}} & 1 & 1+\frac{p_{13}}{p_{11}} & 1 & 1 \\ 0 & 0 & 1+\frac{p_{12}}{p_{11}} & 1 & 1+\frac{p_{13}}{p_{11}} & 1 & 1 \end{array}\right]$$

$$\left(\xrightarrow{R3=\frac{R3}{1+\frac{p_{12}}{p_{11}}}}\right)$$

$$\left[\begin{array}{cccccc|c} 1 & 1 & -\frac{p_{12}}{p_{11}} & 0 & -\frac{p_{13}}{p_{11}} & 0 & 0 \\ 0 & 1 & 0 & \frac{p_{12}}{p_{11}} & -\frac{p_{13}}{p_{11}} & -\frac{p_{13}}{p_{11}} & 0 \\ 0 & 0 & 1 & \frac{p_{11}}{p_{11}+p_{12}} & \frac{p_{11}+p_{13}}{p_{11}+p_{12}} & \frac{p_{11}}{p_{11}+p_{12}} & \frac{p_{11}}{p_{11}+p_{12}} \\ 0 & 0 & \frac{p_{11}+p_{12}}{p_{11}} & 1 & \frac{p_{11}+p_{13}}{p_{11}} & 1 & 1 \end{array}\right]$$

$$\left(\xrightarrow{R4=R4-\frac{(p_{11}+p_{12})}{p_{11}}R3}\right)$$

$$\left[\begin{array}{cccccc|c} 1 & 1 & -\frac{p_{12}}{p_{11}} & 0 & -\frac{p_{13}}{p_{11}} & 0 & 0 \\ 0 & 1 & 0 & \frac{p_{12}}{p_{11}} & -\frac{p_{13}}{p_{11}} & -\frac{p_{13}}{p_{11}} & 0 \\ 0 & 0 & 1 & \frac{p_{11}+p_{12}}{p_{11}} & \frac{p_{11}+p_{12}}{p_{11}+p_{13}} & \frac{p_{11}}{p_{11}+p_{12}} & \frac{p_{11}}{p_{11}+p_{12}} \\ 0 & 0 & 0 & 1-\left(\frac{p_{11}+p_{12}}{p_{11}}\right)^2 & \frac{(p_{11}+p_{13})^2-(p_{11}+p_{12})^2}{p_{11}(p_{11}+p_{13})} & 0 & 0 \end{array}\right]$$

$$\left(\xrightarrow{R5=\frac{R5}{1-(\frac{p_{11}+p_{12}}{p_{11}})^2}}\right)$$

$$\left[\begin{array}{cccccc|c} 1 & 1 & -\frac{p_{12}}{p_{11}} & 0 & -\frac{p_{13}}{p_{11}} & 0 & 0 \\ 0 & 1 & 0 & \frac{p_{12}}{p_{11}} & -\frac{p_{13}}{p_{11}} & -\frac{p_{13}}{p_{11}} & 0 \\ 0 & 0 & 1 & \frac{p_{11}+p_{12}}{p_{11}} & \frac{p_{11}+p_{12}}{p_{11}+p_{13}} & \frac{p_{11}}{p_{11}+p_{12}} & \frac{p_{11}}{p_{11}+p_{12}} \\ 0 & 0 & 0 & 1 & \alpha_3 & 0 & 0 \end{array}\right]$$

where $\alpha_3 = \frac{\frac{(p_{11}+p_{13})^2-(p_{11}+p_{12})^2}{p_{11}(p_{11}+p_{13})}}{1-(\frac{p_{11}+p_{12}}{p_{11}})^2} = \frac{2p_{11}(p_{13}-p_{12})-p_{12}^2+p_{13}^2)}{p_{11}(p_{11}+p_{13})} * \frac{-p_{11}^2}{2p_{11}p_{12}+p_{12}^2}$

Row 4 of the system implies that $y_{23}+\alpha_3 * y_{31} = 0$. If $\alpha_3 > 0 \implies y_{23} < 0 \Rightarrow\!\Leftarrow$

If $\alpha_3 < 0 \implies y_{21} < 0 \Rightarrow\!\Leftarrow$

If $\alpha_3 = 0 \implies y_{23} = 0 \implies y_{12} < 0 \Rightarrow\!\Leftarrow$

$\implies$ This system of equations is inconsistent and therefore has no solution. Invoking Farkas' Lemma, this implies that our original system of equations has a solution.

$\implies \exists \; \{w_{i,j}\}'s$ such that the conditions for a CE are satisfied. $\square$

6 Conclusion

In this paper, I delve into the behavior of Vijay Krishna's conjecture in small cases and attempt to find patterns to inform an induction argument. In order to motivate this research properly, I define both Nash Equilibria and Correlated Equilibria. I also define some of the tools in convex optimization and linear programming that were used in observing the behavior of the conjecture of interest.

The two major approaches in attempting to inform an induction argument were constraint relaxation of the ALT and reducing a game of size 2×3 to games of size 1×3 and 2×2 in an attempt to show infeasibility by contradiction. Although neither approach led to tractability for induction, hopefully they will spur further research into this conjecture. In the game reduction approach (2×3 case) I used the facts that all 1×3 and 2×2 cases are infeasible. To this end, I proved these claims computationally using Gaussian elimination that the conjecture was true for these cases. I observed the same game reduction approach to square cases of size 3 since to take full advantage of symmetry. However, this attempt also failed.

In conclusion, this paper works to push game theory towards understanding of the behavior of Vijay Krishna's conjecture, which is philosophically thoughtprovoking since we often can only observe players' actions but may not have access to the information of the game in which they are participating. This conjecture allows us not to reverse engineer the exact game with certainty, but to create a game for which they are playing optimally with respect to rational play.

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